Post-Newtonian Expansion of Gravitational Radiation[*]

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The post-Newtonian expansion appears to be a relevant tool for predicting the gravitational waveforms generated by some astrophysical systems such as binaries. In particular, inspiralling compact binaries are well-modelled by a system of two point-particles moving on a quasi-circular orbit whose decay by emission of gravitational radiation is described by a post-Newtonian expansion. In this paper we summarize the basics of the computation by means of a series of multipole moments of the exterior field generated by an isolated source in the post-Newtonian approximation. This computation relies on an ansatz of matching the exterior multipolar field to the inner field of a slowly-moving source. The formalism can be applied to point-particles at the price of a further ansatz, that the infinite self-field of point-particles can be regularized in a certain way. As it turns out, the concept of point-particle requires a precise definition in high post-Newtonian approximations of general relativity.

I. INTRODUCTION

Binary systems of compact objects (neutron stars or black holes) emitting gravitational waves in their late stage of evolution leading to a final coalescence might play the major role in the XXIth-century gravitational-wave astronomy. In the final stage, the two objects collide; for instance two black holes form a single black hole which rings down and emits gravitational waves before settling down into a stationary configuration. We refer to the papers in this volume devoted to numerical relativity [1] and to the close-limit approximation [2] for descriptions of the coalescence phase.

In the earlier phase, preceding immediately the coalescence, the two objects undergo a long (adiabatic) inspiral driven by the emission of gravitational radiation, or equivalently by the radiation reaction forces applied to the orbit. In this paper we are mainly interested in the inspiral phase of compact binaries. During this phase the gravitational radiation is essentially produced by the dynamical motion of the two compact objects. In principle, the dynamics can be well-approximated by a post-Newtonian expansion of general relativity. Given the highly relativistic nature of inspiralling compact binaries (the orbital velocity can reach 30% of the speed of light in the last rotations), the problem is just that of pushing the post-Newtonian approximation farther enough in order to be useful to future observations. In recent years it has been realized by several groups [3–10] that in the case of inspiralling neutron star binaries the post-Newtonian expansion should be controlled up to the very high 3PN order. During the inspiral, the internal structure of the stars plays a little role, and one can conveniently describe the two compact bodies by "point-particles" (what only matters are the two masses, but apart from that the objects can be ordinary neutron stars, or black holes or even naked singularities). Note that a priori the concept of point-particle does not make sense in general relativity except in the test-mass limit. However within a post-Newtonian approximation one can give a sense to what we call a point-particle; but even there, particularly when going to high post-Newtonian orders, the concept of a point-particle is non-trivial and must be carefully defined.

A post-Newtonian computation of the inspiral of compact binaries can be based on the following strategy. One starts by implementing a general formalism for the dynamics and the gravitational-wave emission of a slowly-moving isolated source. By slowly-moving we mean the existence of a small post-Newtonian parameter, say

$$\varepsilon = \text{Max}\left\{\left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right|^{1/2}\right\}, \tag{1}$$

where $T^{\mu\nu}$ denotes the matter stress-energy tensor in some Cartesian coordinate system covering the source;[1] Greek indices take values 0,1,2,3, and Latin 1,2,3. We allow ε to be as large as 30% in order to cover the case of inspiralling

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[1]We assume that the source is self-gravitating, so that $\varepsilon \sim (GM/ac^2)^{1/2}$ where M and a denote the mass and radius.

compact binaries. As we see, strictly speaking the source is "slowly-moving", but in fact it can be quite relativistic. As usual, neglecting terms in the gravitational field according to their formal order in ε is valid in the so-called near-zone of the source; the region for which $r/\lambda = O(\varepsilon)$, where λ is a typical wavelength of the radiation field.

The crucial demand on any wave-generation formalism is to be able to relate the gravitational waveform far from the source (to order $1/R$ in the distance to the source at retarded times) to the matter stress-energy tensor $T^{\mu\nu}$. This problem of relating the retarded far-field to the source's matter content is extremely difficult within the exact theory, because of the non-linearities of the field equations; however, the solution exists in a framework of post-Newtonian approximations for slowly-moving sources. Another requirement for a general formalism is for its ability to control the equations of motion of the source, and in particular the gravitational radiation reaction forces therein. The radiation reaction forces are to be consistent with the radiation field at infinity (by definition), and thus to depend on boundary conditions such as the no-incoming radiation condition imposed at past null infinity (which ensures that the source is physically isolated). In a post-Newtonian approximation, a difficulty of the problem is that the radiation reaction forces enter the equations of motion of the source which are determined by the post-Newtonian expansion valid only in the near zone. Therefore one must supplement the post-Newtonian expansion by a condition of matching the near-zone field to the radiation field [see (2·5) below]. In this article we outline a particular post-Newtonian formalism, making extensive use of multipole moments, which is issued from work of Blanchet and Damour, [11–13] Damour and Iyer, [14] and Blanchet. [16–18] A different formalism recently defined by Will and Wiseman [19] on foundations laid by Epstein and Wagoner [20] and Thorne [21] is described by C. Will in this volume.

Note that the present post-Newtonian formalism is interested in the *formal* post-Newtonian expansion $\varepsilon \to 0$; in particular it does not try to investigate the exact mathematical nature of the post-Newtonian series. Thus the connection between the approximation and the exact theory is not controlled.[2] Simply, one looks for a mathematically well-defined scheme for generating the successive post-Newtonian approximations. The hope is that in applications, when comparing to observations the post-Newtonian prediction developed to high enough order in ε, we shall drastically limit the magnitude of systematic errors, resulting in a very accurate measurement.

Be aware that at present no systematic algorithm for generating the post-Newtonian series of the near-zone gravitational field exists. One can express the multipole moments of the source in terms of the infinite formal post-Newtonian series, [18] but one does not know how to compute *explicitly* this series to all orders, in the manner, say, of the explicit algorithm for generating the post-*Minkowskian* series in the source's exterior. [11] But of course, the first few post-Newtonian approximations, up to, say, the 2.5PN order, are very well understood. [23–28] A difficulty associated with the post-Newtonian iteration is the appearance in higher post-Newtonian orders of Poisson-type integrals which are divergent at the bound at infinity. This comes from the fact that the post-Newtonian iteration is valid only in the near-zone, and so the post-Newtonian coefficients rapidly blow up far from the system. This signals that the standard Poisson integral does not constitute the relevant solution of the Poisson equation in this context.

A crucial element of the present formalism is that it is *a priori* only valid for continuous (in fact, smooth) matter distributions. Thus, $T^{\mu\nu}$ is assumed from the start to be regular, for instance to describe a smooth hydrodynamical fluid. This excludes *a priori* the very interesting application to "self-gravitating" point-particles (as opposed to "test" point-particles), whose self-field becomes infinite at the location of the particle, thus creating a singularity. So we are obliged to introduce a new ingredient. Our proposal is that this be a regularization *à la* Hadamard [29,30] for removing systematically the infinite self-field of the particles. Only when we assume a regularization, can we use a $T^{\mu\nu}$ constituted of Dirac delta-functions. The removing of the divergent terms is for the moment done without further justification; simply this is an ansatz which, as far as we can see, yields consistent computations in practice, and which has been checked to yield the correct result in some cases, but which remains an ansatz. Notably we do not prove that the regularization is still permissible in higher post-Newtonian approximations, or even that it is possible to find a consistent regularization at all, or that the result of two different (though consistent) regularizations would be the same. We will content ourselves with the use of the Hadamard regularization which yields in practice, to the first few post-Newtonian orders, some consistent (and rather elegant) computations.

A different strategy is possible when we have at our disposal a natural background space-time. This is the case when the mass ratio of two particles is so small that one can view one of them as moving in the Schwarzschild or Kerr background generated by the other. In the small mass-ratio limit the particle moves on a geodesic of the background, and we can compute the emitted radiation using a linear background perturbation. This approach has reached a mature state: notably the gravitational radiation emitted by a test particle in orbit around a Schwarzschild black hole was computed to very high post-Newtonian order; [31–33] this computation represents an important benchmark against which the standard post-Newtonian expansion can be checked. To second-order in the black-hole perturbation

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[2]A mean to understand the limit relation of Einstein's theory to Newton's is to introduce a frame theory *à la* Ehler. [22]

there are terms vanishing in the mass-ratio limit, the particle reacts to the gravitational radiation emission, and again we face the problem of how to regularize the particle's self-field.

II. GENERAL SOURCES

For a general compact-support stress-energy tensor $T^{\mu\nu}$, we want to solve the field equations of general relativity in the form of a post-Newtonian expansion $\varepsilon \to 0$. We reduce the field equations by means of the condition of harmonic coordinates, that is $\partial_\nu h^{\mu\nu} = 0$ where $h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$, so that

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}(h, \partial h, \partial^2 h) \ , \tag{2}$$

where we have introduced a Minkowskian background, $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1) = \eta_{\mu\nu}$ (with respect to which all indices are raised and lowered), and the associated d'Alembertian operator, $\Box \equiv \Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The gravitational source term $\Lambda^{\mu\nu}$ is a complicated functional of the field and its first and second space-time derivatives. We define the total stress-energy (pseudo-)tensor $\tau^{\mu\nu}$ of the matter and gravitational fields as

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu} \ . \tag{3}$$

Of course $\tau^{\mu\nu}$ is not a generally-covariant tensor, but it is a Lorentz tensor relatively to our Minkowskian background. It is conserved in a Lorentz-covariant sense, and this is equivalent to the covariant conservation of $T^{\mu\nu}$,

$$\partial_\nu \tau^{\mu\nu} = 0 \quad \Leftrightarrow \quad \nabla_\nu T^{\mu\nu} = 0 \ . \tag{4}$$

The propagation of $h^{\mu\nu}$ subject to the Einstein field equations (2·1) is a well-posed problem, for which we need to choose some initial conditions in the past. In this paper we shall assume that the field is stationary before some remote date $-\mathcal{T}$, so that there is no radiation generated by sources at infinity incoming onto the system. Arguably, the condition of stationarity in the past is too strong; for instance it does not cover a physical situation where two bodies moving initially on unbound (hyperbolic-like) orbits would form a bound system. However one may in some cases justify the assumption *a posteriori*, by checking that the formulas obtained under it are still valid in a more general physical situation such as the initial scattering of two bodies. With no-incoming radiation one can transform the differential Einstein equations (2·1) into the integro-differential equations

$$h^{\mu\nu} = \frac{16\pi G}{c^4}\Box_R^{-1}\tau^{\mu\nu} \equiv -\frac{4G}{c^4}\int \frac{d^3\boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|}\tau^{\mu\nu}\left(\boldsymbol{x}', t - |\boldsymbol{x} - \boldsymbol{x}'|/c\right) \ , \tag{5}$$

where $\Box_R^{-1}$ denotes the standard retarded inverse d'Alembertian.

In our approach we resolve the wave-generation problem by finding the solution of the field equations (2·4) in the *exterior* region of the source (outside the compact support of $T^{\mu\nu}$) by means of an infinite multipole-moment series for $h^{\mu\nu}$, that we denote $\mathcal{M}(h^{\mu\nu})$. This means in particular that we must be able to relate the multipole moments parametrizing this series to the matter content of the source. In general this is not an easy task, but this can be done using the post-Newtonian expansion, in the physical case where the source is slowly-moving. The latter source multipole moments are then "propagated" with the help of a post-Minkowskian expansion to large distances from the source, and related there to the so-called radiative multipole moments directly accessible to a far-away observer.[3]

To obtain the expression of the multipole moments of the source in terms of $T^{\mu\nu}$ we use an asymptotic matching between the multipole expansion $\mathcal{M}(h^{\mu\nu})$, valid everywhere outside the source, and the post-Newtonian expansion denoted $\overline{h}^{\mu\nu}$, valid in the near-zone. For slowly-moving sources the two domains of validity of the multipole and post-Newtonian expansions overlap in the so-called exterior near-zone, and we can write there the numerical equality $\mathcal{M}(h^{\mu\nu}) = \overline{h}^{\mu\nu}$. Then we transform this equality into a "matching equation", that is an equation between two series of the same nature, which is formally valid "everywhere". For this purpose we replace the multipole terms on the left-hand side of the equality by their formal post-Newtonian expansions (this means expanding all retardations $t - r/c$ when $c \to \infty$; so in fact the latter expansion is equivalent to a near-zone expansion $r \to 0$), and we consider on the

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[3]Using a post-Minkowskian expansion for the exterior field in conjunction with the multipolar series is an old idea of Bonnor, [34] later generalized by Thorne. [21]

right-hand side the multipole decomposition of (each of the coefficients of) the near-zone expansion. Therefore the matching equation reads

$$\overline{\mathcal{M}(h^{\mu\nu})} = \mathcal{M}(\overline{h}^{\mu\nu}) \ . \tag{6}$$

Satisfying this equation permits in principle to find the unique physical solution of the field equations valid inside and outside the source (and, in the present case, in the uniquely defined harmonic coordinate system).

Let us simply announce here the result [16,18] for the expression of $\mathcal{M}(h^{\mu\nu})$ satisfying at once the vacuum Einstein field equations (with no-incoming radiation) and the matching equation (2·5). There are two terms,

$$\mathcal{M}(h^{\mu\nu}) = -\frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left\{ \frac{1}{r} \mathcal{F}_L^{\mu\nu}(t - r/c) \right\} + \mathrm{FP}_{B=0} \, \Box_R^{-1} [r^B \mathcal{M}(\Lambda^{\mu\nu})] \ . \tag{7}$$

The second term constitutes a *particular* solution of the vacuum field equations, which is defined by analytic continuation in a complex parameter B as being the finite part of the Laurent expansion when $B \to 0$ (in our notation $\mathrm{FP}_{B=0}$). The reason for the need of a finite part is that the integrand of the retarded integral is in the form of a multipole expansion which is not valid inside the source, and which is actually singular at the spatial origin of the coordinates $r = 0$ located inside the source. Now the first term in (2·6) (in which we denote $L = i_1 \cdots i_l$ and $\partial_L \equiv \partial_{i_1} \cdots \partial_{i_l}$) represents clearly an *homogeneous* solution of the wave equation. This solution is uniquely singled out so that when added to the particular solution it ensures the satisfaction of the matching equation (2·5). The "multipole moments" parametrizing this solution are given as [18]

$$\mathcal{F}_L^{\mu\nu}(u) = \mathrm{FP}_{B=0} \int d^3\boldsymbol{x} \ |\boldsymbol{x}|^B \hat{x}_L \int_{-1}^1 dz \ \delta_l(z) \overline{\tau}^{\mu\nu}(\boldsymbol{x}, u + z|\boldsymbol{x}|/c), \tag{8}$$

(where $u = t - r/c$). The moments are also defined by analytic continuation in B, which makes them well-defined mathematically (note that the integrand behaves typically as a positive power of the distance at infinity, and thus that the integral would be strongly divergent at infinity without any finite part). In (2·7) we use a special notation for a symmetric trace-free product of vectors: $\hat{x}_L \equiv \mathrm{STF}(x_L)$ where $x_L \equiv x_{i_1} \cdots x_{i_l}$ (we have $\delta_{i_l i_{l-1}} \hat{x}_L = 0$). The z-integration in (2·7) involves the weighting function

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1} l!} (1 - z^2)^l \ ; \quad \int_{-1}^1 dz \ \delta_l(z) = 1 \ . \tag{9}$$

In the limit of large l we have $\lim_{l \to \infty} \delta_l = \delta$ the Dirac measure.

The crucial point about the multipole moments (2·7) is that they are generated by the *post-Newtonian* expansion of the stress-energy pseudo-tensor, that is $\overline{\tau}^{\mu\nu}$, rather than $\tau^{\mu\nu}$ itself. It is at this point that our assumption of matching with a slowly-moving post-Newtonian source enters. Note that although the integrand of the multipole moments is in the form of a post-Newtonian expansion valid only in the near-zone, the integration is to be performed on the whole 3-dimensional space (see Ref. [18] for details). This is one of the beauties of the analytic continuation, that it permits to handle integrals over $I\!\!R^3$ without introducing a cutoff at the edge of the near-zone. On the contrary, Will and Wiseman [19] do not use analytic continuation, and consider integrals extending only over the near-zone. It can be shown that the present formalism is equivalent to that of Will and Wiseman.

With the expression (2·7) in hands, it is straightforward to define six sets of irreducible multipole moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ associated with the six independent components of $\mathcal{F}_L^{\mu\nu}$ (ten minus four because of the harmonic coordinate condition). This constitutes our primary definition for the source multipole moments. With more work (considering a coordinate transformation in the exterior zone) we can further define only two sets of irreducible moments $\{M_L, S_L\}$ which are less directly connected to the source but are very useful in practical computations.

Convenient notions of the source multipole moments being chosen, let us relate them to the radiative moments at infinity. We follow the standard definition [21] that the radiative moments parametrize in radiative (Bondi-type) coordinates the leading term $1/R$ in the distance to the source. Clearly, looking at (2·6), we see that the radiative moments have two contributions. Essentially the contribution coming from the first term of (2·6) is that of the source multipole moments $I_L, J_L, \cdots, Z_L$. In a linear theory the radiative moments would contain only this contribution, i.e., they would agree with the source moments. But in general relativity we have also the contribution of the second term in (2·6) (which is at least quadratic in the field strength G), so there will be many non-linear interactions between the source moments to any order in G. Therefore the radiative moments are given by some (very complicated) non-linear functionals of the source moments. We compute these with the help of a post-Minkowskian algorithm. [11,15] That is, we are able to rewrite (2·6) in the form of a formal expansion when $G \to 0$,

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$$\mathcal{M}(h^{\mu\nu}) = G h_1^{\mu\nu}[I_L, J_L, \cdots] + G^2 h_2^{\mu\nu} + \cdots + G^n h_n^{\mu\nu} + \cdots \ , \tag{10}$$

where the first term is a solution of the linearized field equations which depends on the source moments [this is essentially the first term in (2·6)], and where all the subsequent non-linear corrections are constructed by post-Minkowskian iteration (see Ref. [11] for the proof that this can be done to any post-Minkowskian order). Next we change coordinates from harmonic to radiative ones $(T, \boldsymbol{X})$ and expand the metric when $R \to \infty$. All the physical information about the radiation field is contained into the so-called transverse-traceless (TT) projection of the spatial (ij) metric. The radiative moments $\{U_L, V_L\}$ are defined from the term of order $1/R$ as

$$\mathcal{M}(h_{ij})^{\mathrm{TT}}(\boldsymbol{X}, T) = -\frac{4G}{c^2 R} \mathcal{P}_{ijab} \sum_{l \geq 2} \frac{1}{c^l l!} \left\{ N_{L-2} U_{abL-2} - \frac{2l}{c(l+1)} N_{cL-2} \varepsilon_{cd(a} V_{b)dL-2} \right\}$$
$$+ O\left(\frac{1}{R^2}\right) \ , \tag{11}$$

where $N_i = X^i/R$, $N_{L-2} = N_{i_1} \cdots N_{i_{l-2}}$, etc., and where the TT projector reads $\mathcal{P}_{ijab} = (\delta_{ia} - N_i N_a)(\delta_{jb} - N_j N_b) - \frac{1}{2}(\delta_{ij} - N_i N_j)(\delta_{ab} - N_a N_b)$. Now it remains to compare the result of the iteration of the field (2·9) with the relation (2·10) in order to deduce the radiative moments U_L and V_L in terms of the source moments, and therefore in terms of $T^{\mu\nu}$ via the post-Newtonian expansion of the pseudo-tensor $\tau^{\mu\nu}$ [see (2·7)].

The wave-generation is complete in this way. Note that the formalism is general in the sense that we did not specify a particular form of $T^{\mu\nu}$. Thus, for application to a particular problem, we need first to choose a model of $T^{\mu\nu}$, for instance a perfect fluid or a model of point-particles (see below), and insert that model into the formalism.

III. POINT-PARTICLES

The choice of a model of point-particles in (post-Newtonian) general relativity is non-trivial because it is intimately related with the choice of a regularization for removing the self-field of point-particles. Of course, for a *test* particle moving on a fixed smooth gravitational background $g_{\mu\nu}^B$, there is no problem and the stress-energy tensor reads

$$T_{\mathrm{test}}^{\mu\nu}(\boldsymbol{x}, t) = \frac{m v^\mu v^\nu}{\sqrt{-g_{\rho\sigma}^B(y) v^\rho v^\sigma/c^2}} \frac{\delta(\boldsymbol{x} - \boldsymbol{y}(t))}{\sqrt{-g^B(x)}} \ , \tag{12}$$

where $v^\mu = dy^\mu/dt$ is the particle's coordinate velocity and δ is the 3-dimensional Dirac measure (a particular case of distribution).

However, in the case of "*self-gravitating*" particles contributing to the gravitational field, the previous stress-energy tensor does not make sense. Indeed, already at the Newtonian order, the metric $g_{\mu\nu}$ generated by two particles for instance contains the Newtonian potential $U = Gm_1/r_1 + Gm_2/r_2$ where $r_{1,2} = |\boldsymbol{x} - \boldsymbol{y}_{1,2}|$ is the distance between the field point and the source points. Clearly U is infinite at the location of the particles, and must be regularized in some way in order to speak of its value at point 1, say $(U)_1$. We know that the correct result in Newtonian gravity is $(U)_1 = Gm_2/r_{12}$ where $r_{12} = |\boldsymbol{y}_1 - \boldsymbol{y}_2|$. But in high post-Newtonian orders we shall meet more difficult quantities such as $(U^4)_1$ which necessitate a precise definition.

Let us model the stress-energy tensor of two point-particles in post-Newtonian approximations of general relativity by

$$T^{\mu\nu}(\boldsymbol{x}, t) = \frac{m_1 v_1^\mu v_1^\nu}{\sqrt{-(g_{\rho\sigma})_1 v_1^\rho v_1^\sigma/c^2}} \frac{\delta(\boldsymbol{x} - \boldsymbol{y}_1(t))}{\sqrt{-g(x)}} + 1 \leftrightarrow 2 \ , \tag{13}$$

where $1 \leftrightarrow 2$ means the exchange of 1 to 2 on the two previous terms and, $(g_{\rho\sigma})_1$ denotes the *regularized* value of the metric $g_{\rho\sigma}$ at the location of the particle 1, according to a procedure based on the Hadamard [29,30] partie finie of a singular function and a divergent integral. Note that in fact the chosen regularization *defines* our model of point-particles. A conjecture would be that another type of regularization, if sufficiently powerful to produce unambiguous results at high post-Newtonian orders (as seems to be the case of the Hadamard regularization), would yield identical physical results.

To define the Hadamard regularization we introduce an appropriate class of functions $\mathcal{F}$ on $\mathbb{R}^3$. By definition $F(\boldsymbol{x})$ belongs to $\mathcal{F}$ if and only if it is smooth on $\mathbb{R}^3$ except at two singular points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, around which it admits a power-like expansion of the type

$$F = \sum_a r_1^a f_{1(a)}(\boldsymbol{n}_1) \qquad \text{when } r_1 \to 0 \tag{14}$$

(and idem for 2). The summation index a is assumed to be bounded from below, $a \geq -a_0$ (no essential singularity), and to take real discrete values, say $a \in \{a_j\}_{j \in \mathbb{N}}$ where $a_j \in \mathbb{R}$. In fact in most cases one can assume $a \in \mathbb{Z}$. As indicated the coefficients $f_{1(a)}$ depend on the unit direction $\boldsymbol{n}_1$ of approach to the singularity. In (3·3) we do not write any remainder for the expansion because we shall need only formulas depending on the $f_{1(a)}$, and never the complete expansion itself. [4]

First we define the Hadamard partie finie of the singular function F at the location of the singularity 1 for instance, say $(F)_1$. Simply we pick up the coefficient of the zeroth power of r_1 in (3·3), namely $f_{1(0)}$, and average over all directions $\boldsymbol{n}_1$:

$$(F)_1 \equiv \int \frac{d\Omega_1}{4\pi} f_{1(0)} \ . \tag{15}$$

Second we define the partie finie (in short Pf) of the divergent integral $\int d^3\boldsymbol{x} \ F$. We remove from $\mathbb{R}^3$ two spherical balls surrounding the two singularities, of the form $r_{1,2} \leq s$, where s is a small radius. Using (3·3) it is easy to determine the expansion when $s \to 0$ of the integral extending on $\mathbb{R}^3$ deprived from these two balls, next to subtract all the terms which are divergent when $s = 0$, and then to take the limit $s \to 0$ of what remains. The result is the Hadamard partie finie. Actually we find some logarithms of s in the expansion, so there is an ambiguity linked with the freedom of choosing a unit of length to measure s. To account with this ambiguity we introduce a constant length scale s_1 (and similarly s_2). We find

$$\text{Pf} \int d^3\boldsymbol{x} \ F \equiv \lim_{s \to 0} \left\{ \int_{\substack{r_1 > s \\ r_2 > s}} d^3\boldsymbol{x} \ F + \sum_{a+3 \leq -1} \frac{s^{a+3}}{a+3} \int d\Omega_1 f_{1(a)} \right.$$
$$\left. + \ln\left(\frac{s}{s_1}\right) \int d\Omega_1 f_{1(-3)} + 1 \leftrightarrow 2 \right\} , \tag{16}$$

The two definitions (3·4) and (3·5) are closely related to each other. To see this, apply (3·5) to the case where the function is actually a gradient $\partial_i F$ (it is clear that $F \in \mathcal{F}$ implies $\partial_i F \in \mathcal{F}$). We find

$$\text{Pf} \int d^3\boldsymbol{x} \ \partial_i F = -4\pi(n_1^i r_1^2 F)_1 - 4\pi(n_2^i r_2^2 F)_2 , \tag{17}$$

whose proof involves the Gauss theorem on the two surfaces $r_{1,2} = s$ surrounding the two singularities. Thus, for singular functions, the integral of a gradient is not zero. This indicates that the "ordinary" derivative is not adequate for the purpose of application of a "fluid" formalism to point-particles (because the integral of a gradient is always zero for fluids). Instead one must generalize the derivative to take into account the singularities, in a way similar to the distributional derivative of distribution theory. [30]

Another nice connection between the two definitions (3·4) and (3·5) is

$$\lim_{\epsilon \to 0} \text{Pf} \int d^3\boldsymbol{x} \ \delta_\epsilon(\boldsymbol{x} - \boldsymbol{y}_1) F(\boldsymbol{x}) = (F)_1 , \tag{18}$$

where δ_ϵ denotes the Riesz [45] delta-function defined for any $\epsilon > 0$ by

$$\delta_\epsilon(\boldsymbol{x}) = \frac{\epsilon(1-\epsilon)}{4\pi} |\boldsymbol{x}|^{\epsilon-3} , \qquad \text{so that} \qquad \Delta|\boldsymbol{x}|^{\epsilon-1} = -4\pi\delta_\epsilon \ . \tag{19}$$

Clearly, in the limit $\epsilon \to 0$, the Riesz delta-function yields a generalization of the Dirac measure applicable to the Hadamard partie finie of a singular function in $\mathcal{F}$.

To conclude, let us give the result of the application of the general formalism to a binary system of point-particles, concerning the total flux (or "gravitational luminosity" $\mathcal{L}$) emitted by the binary in the form of gravitational waves. This quantity plays a crucial role in the computation of the orbital phase of the binary, as it evolves with time taking

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[4]In addition to (3·3) we assume that the functions $F \in \mathcal{F}$ decrease sufficiently rapidly when $|\boldsymbol{x}| \to \infty$, so that all integrals we consider are convergent at infinity.

into account the loss of energy by gravitational waves. For circular orbits the orbital phase $\phi = \int \omega dt$ is obtained from the energy balance equation as

$$\frac{dE}{dt} = -\mathcal{L} \qquad \Rightarrow \qquad \phi = -\int \frac{\omega dE}{\mathcal{L}} \, , \tag{20}$$

where E denotes the orbital binding energy of the binary in the center of mass frame. Let m_1 and m_2 be the two masses, and denote $m = m_1 + m_2$ and the mass ratio $\nu = \mu/m = m_1 m_2/m^2$. As a small post-Newtonian parameter we define $x = (Gm\omega/c^3)^{2/3}$ where $\omega = 2\pi/P$ is the orbital frequency and P the period [x is of order $O(\varepsilon^2)$ in the notation (1·1)]. The present post-Newtonian accuracy for $\mathcal{L}$ is 3.5PN except that the contributions proportional to the mass ratio ν in the 3PN term are not yet under control. These contributions are indicated by $O(\nu)$ in the formula below; their computation is a work in progress (collaboration with Faye, Iyer and Joguet). We obtain

$$
\begin{aligned}
\mathcal{L} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} \\
& + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 \\
& + \left(-\frac{8191}{672} - \frac{535}{24}\nu\right)\pi x^{5/2} \\
& + \left(\frac{6643739519}{69854400} - \frac{1712}{105}C - \frac{856}{105}\ln(16x) + \frac{16}{3}\pi^2 + O(\nu)\right)x^3 \\
& + \left(-\frac{16285}{504} + \frac{176419}{1512}\nu + \frac{19897}{378}\nu^2\right)\pi x^{7/2} + O(x^4) \Bigg\} .
\end{aligned}
\tag{21}
$$

The Newtonian approximation was given by Landau and Lifschitz [35] and the classic work of Peters and Mathews [36] (recall that the Newtonian term is sufficient to account for the radiation emitted by the binary pulsar PSR1913+16). The 1PN term was computed long ago by Wagoner and Will [37] applying the formalism of Epstein and Wagoner, [20] and later confirmed [38] applying the present formalism. The 1.5PN term (with 4π as a factor) is due to gravitational-wave tails, which was first obtained by Poisson [39] using a black-hole perturbation, and it was re-computed by Wiseman [40] and Blanchet and Schäfer [41] within the present formalism. The 2PN term was computed independently by Blanchet, Damour and Iyer [42] applying the same formalism, and by Will and Wiseman [19] applying their new formalism (see Ref. [43] for a summary). The 2.5PN and 3.5PN orders are due to higher-order tail effects. [44,17] The 3PN order is very interesting. It involves a logarithmic term (first obtained in this context by Tagoshi and Nakamura, [7]) as well as the Euler constant $C = 0.577\cdots$ and a π^2. These contributions in the 3PN order, computed in Ref. [17], are due to the so-called tails of tails, that is to say, tails generated by the tails themselves — a purely cubic effect. The corrections $O(\nu)$ in the 3PN term depend on the full details concerning the regularization outlined previously.

Using x as a post-Newtonian parameter is convenient because of its invariant meaning (the same in all coordinate systems). Thus the formula (3·10) can be directly compared with the one obtained using a black-hole perturbation (which typically describes the background in Schwarzschild coordinates) triggered by the motion of a test particle of mass νm in the field of a black-hole of mass m. Using this method Sasaki, Tagoshi and Tanaka [31–33] obtained $\mathcal{L}$ to 5.5PN order in the limit $\nu \to 0$. In the same limit the formula (3·10) agrees completely with these results to the corresponding order. Notably the rational fraction 6643739519/69854400 in the 3PN coefficient comes out exactly the same in the post-Newtonian formalism as in the black-hole perturbation approach (collaboration with Iyer and Joguet).

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